

# CONSTRUCTION OF HYPERBOLIC HYPERSURFACES OF LOW DEGREE IN $\mathbb{P}^n(\mathbb{C})$

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## Abstract

We construct families of hyperbolic hypersurfaces  $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$  of degree  $d \geq (\frac{n+3}{2})^2$ .

**Keywords:** Kobayashi conjecture, hyperbolicity, Brody Lemma, Nevanlinna Theory

## 1 Introduction and the main result

It was conjectured by Kobayashi [12] in 1970 that a generic hypersurface  $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$  of sufficiently high degree  $d \geq d(n) \gg 1$  is hyperbolic. According to Zaidenberg [20], the optimal degree bound should be  $d(n) = 2n + 1$ .

This conjecture, with nonoptimal degree bound in the assumption, was proved, in the case of surface in  $\mathbb{P}^3(\mathbb{C})$ , by Demailly and El Goul [6], and later, by Păun [14] with a slight improvement of the degree bound, and in the case of three-fold in  $\mathbb{P}^4(\mathbb{C})$  [15], [8]. For arbitrary  $n$ , it was proved in [7] that any entire curve in generic hypersurface  $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$  of degree  $d \geq 2n^5$  must be algebraically degenerate. An improvement of the effective degree bound in this result was given in [4]. Recently, for any dimension  $n$ , a positive answer for generic hypersurfaces of degree  $d \geq d(n) \gg 1$  very high was proposed by Siu [18], and a strategy which is expected to give a confirmation of this conjecture for *very* generic hypersurfaces of degree  $d \geq 2n + 2$  was announced by Demailly [5].

Another direction on this subject is to construct examples of hyperbolic hypersurfaces of low degree. In low dimensional case, several examples of hyperbolic hypersurfaces were given. The first example of a hyperbolic surface in  $\mathbb{P}^3(\mathbb{C})$  was constructed by Brody and Green [2]. In  $\mathbb{P}^3(\mathbb{C})$ , Duval [9] gave an example of a hyperbolic surface of degree 6, which is the lowest degree found up to date. Later, Ciliberto and Zaidenberg [3] gave a new construction of hyperbolic surface of degree 6 and their method works for all degree  $d \geq 6$  (hence, this is the first time when a hyperbolic surface of degree 7 was created). In [11], we constructed families of hyperbolic hypersurfaces of degree  $d = d(n) = 2n + 2$  for  $2 \leq n \leq 5$  (the method works for all  $d \geq 2n + 2$ ). The first examples in any dimension  $n \geq 4$  were discovered by Masuda and Noguchi [13], with high degree. Improving this result, examples of hyperbolic hypersurfaces with lower degree asymptotic were given by Siu and Yeung [19] with  $d(n) = 16n^2$ , and by Shiffman and Zaidenberg [16] with  $d(n) = 4n^2$ .

In this note, using the technique of [11], we improve the result of Shiffman and Zaidenberg [16] by proving that a small deformation of a union of  $q \geq (\frac{n+3}{2})^2$  hyperplanes in general position in  $\mathbb{P}^{n+1}(\mathbb{C})$  is hyperbolic.

A family of hyperplanes  $\{H_i\}_{1 \leq i \leq q}$  with  $q \geq n + 1$  in  $\mathbb{P}^n(\mathbb{C})$  is said to be in *general position* if any  $n + 1$  hyperplanes in this family have empty intersection, namely if

$$\cap_{i \in I} H_i = \emptyset, \quad \forall I \subset \{1, \dots, q\}, |I| = n + 1.$$

Let  $\{H_i\}_{1 \leq i \leq q}$  be a family of hyperplanes in general position in  $\mathbb{P}^n(\mathbb{C})$ . A hypersurface  $S$  in  $\mathbb{P}^n(\mathbb{C})$  is said to be in *general position with respect to*  $\{H_i\}_{1 \leq i \leq q}$  if it avoids all intersection points of  $n$  hyperplanes, namely if

$$S \cap \left( \cap_{i \in I} H_i \right) = \emptyset, \quad \forall I \subset \{1, \dots, q\}, |I| = n.$$

**Main Theorem.** Let  $\{H_i\}_{1 \leq i \leq q}$  be a family of  $q \geq (\frac{n+3}{2})^2$  hyperplanes in general position in  $\mathbb{P}^{n+1}(\mathbb{C})$ , where  $H_i = \{h_i = 0\}$ . Then there exists a hypersurface  $S = \{s = 0\}$  of degree  $q$  in general position with respect to  $\{H_i\}_{1 \leq i \leq q}$  such that the hypersurface

$$\Sigma_\epsilon = \{\epsilon s + \Pi_{i=1}^q h_i = 0\}$$

is hyperbolic for sufficiently small complex  $\epsilon \neq 0$ .

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## 2 Preparations

### 2.1 Brody Lemma and its applications

Let  $X$  be a compact complex manifold equipped with a hermitian metric  $\|\cdot\|$ . An *entire curve* in  $X$  is a nonconstant holomorphic map  $f : \mathbb{C} \rightarrow X$ . Such an  $f : \mathbb{C} \rightarrow X$  is called a *Brody curve* if its derivative  $\|f'\|$  is bounded. The following result [1] is a useful tool for studying complex hyperbolicity.

**Brody Lemma.** Let  $f_k : \mathbb{D} \rightarrow X$  be a sequence of holomorphic maps from the unit disk to a compact complex manifold  $X$ . If  $\|f'_k(0)\| \rightarrow \infty$  as  $k \rightarrow \infty$ , then there exist a point  $a \in \mathbb{D}$ , a sequence  $(a_k)$  converging to  $a$  and a decreasing sequence  $(r_k)$  of positive real numbers converging to 0 such that the sequence of maps

$$z \rightarrow f_k(a_k + r_k z)$$

converges toward a Brody curve, after extracting a subsequence.

Consequently, we have a well-known characterization of Kobayashi hyperbolicity.

**Brody Criterion.** A compact complex manifold  $X$  is Kobayashi hyperbolic if and only if it contains no entire curve.

The following form of the Brody Lemma shall be repeatedly used in the proof of the Main Theorem.

**Sequences of entire curves.** Let  $X$  be a compact complex manifold and let  $(f_k)$  be a sequence of entire curves in  $X$ . Then there exist a sequence of reparameterizations  $r_k : \mathbb{C} \rightarrow \mathbb{C}$  and a subsequence of  $(f_k \circ r_k)$  which converges toward an entire curve.

### 2.2 Stability of intersections

We recall here the following known complex analysis fact.

**Stability of intersections.** Let  $X$  be a complex manifold and let  $H \subset X$  be an analytic hypersurface. Suppose that a sequence  $(f_k)$  of entire curves in  $X$  converges toward an entire curve  $f$ . If  $f(\mathbb{C})$  is not contained in  $H$ , then

$$f(\mathbb{C}) \cap H \subset \lim f_k(\mathbb{C}) \cap H.$$

### 2.3 Hyperbolicity of the complement of $2n + 1$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$

We also need the classical generalization of Picard's theorem (case  $n = 1$ ) [10].

**Theorem 2.1.** The complement of a collection of  $2n + 1$  hyperplanes in general position in  $\mathbb{P}^n(\mathbb{C})$  is hyperbolic.

### 3 Proof of the Main Theorem

Given a hypersurface  $S$  of degree  $q$  in general position with respect to the family  $\{H_i\}_{1 \leq i \leq q}$ , we would like to determine what conditions  $S$  should satisfy for  $\Sigma_\epsilon$  to be hyperbolic. Suppose that  $\Sigma_{\epsilon_k}$  is not hyperbolic for a sequence  $(\epsilon_k)$  converging to 0. Then we can find entire curves  $f_{\epsilon_k}: \mathbb{C} \rightarrow \Sigma_{\epsilon_k}$ . By the Brody Lemma, after reparametrization and extraction, we may assume that the sequence  $(f_{\epsilon_k})$  converges to an entire curve  $f: \mathbb{C} \rightarrow \cup_{i=1}^q H_i$ . By uniqueness principle, the curve  $f(\mathbb{C})$  lands in  $\cap_{i \in I} H_i$ , for some subset  $I$  of the index set  $\mathbf{Q} := \{1, \dots, q\}$  and does not land in any  $H_j$  with  $j \in \mathbf{Q} \setminus I$ .

**Lemma 3.1.** One has

$$|I| \leq n - 1.$$

*Proof.* If on the contrary  $|I| = n$ , then for all  $j \in \mathbf{Q} \setminus I$ , by stability of intersections, one has

$$f(\mathbb{C}) \cap H_j \subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \subset \lim \Sigma_{\epsilon_k} \cap H_j \subset S \cap H_j.$$

Thus,  $f(\mathbb{C}) \cap H_j \subset S \cap H_j \cap (\cap_{i \in I} H_i) = \emptyset$ . Hence,  $f(\mathbb{C}) \subset \cap_{i \in I} H_i \setminus (\cup_{j \in \mathbf{Q} \setminus I} H_j)$ , which is a contradiction, since the complement of  $q - |I| > 3$  points in a line is hyperbolic by Picard's theorem.  $\square$

By the above argument,  $f(\mathbb{C}) \cap H_j$  is contained in  $S$  for all  $j \in \mathbf{Q} \setminus I$ . Therefore, the curve  $f(\mathbb{C})$  lands in

$$\cap_{i \in I} H_i \setminus (\cup_{j \in \mathbf{Q} \setminus I} H_j \setminus S). \quad (3.1)$$

So, the problem reduces to finding a hypersurface  $S$  of degree  $q$  such that all complements of the form (3.1) are hyperbolic, where  $I$  is an arbitrary subset of  $\mathbf{Q}$  having cardinality at most  $n - 1$ .

Such a hypersurface  $S$  will be constructed by using the deformation method of Zaidenberg and Shiffman [17].

*Starting point of the deformation process.* Let  $\{H_i\}_{1 \leq i \leq q}$  be a family of hyperplanes in general position in  $\mathbb{P}^n(\mathbb{C})$ . For some integer  $0 \leq k \leq n - 1$  and some subset  $I_k = \{i_1, \dots, i_{n-k}\}$  of the index set  $\{1, \dots, q\}$  having cardinality  $n - k$ , the linear subspace  $P_{k, I_k} = \cap_{i \in I_k} H_i \simeq \mathbb{P}^k(\mathbb{C})$  will be called a *subspace of dimension  $k$* . We will denote by  $P_{k, I_k}^*$  the complement  $P_{k, I_k} \setminus (\cup_{i \notin I_k} H_i)$ , which we will call a *star-subspace of dimension  $k$* . The process of constructing  $S$  by deformation will start with the following result, which is an application of Theorem 2.1.

**Starting Lemma.** Let  $\{H_i\}_{1 \leq i \leq q}$  be a family of  $q \geq (\frac{n+3}{2})^2$  hyperplanes in general position in  $\mathbb{P}^{n+1}(\mathbb{C})$ . Let  $I$  and  $J$  be two disjoint subsets of the index set  $\{1, \dots, q\}$  such that  $1 \leq |I| \leq n - 1$ , and  $|J| = q + m + 1 - 2|I|$  with some  $0 \leq m \leq |I| - 1$ . Then all complements of the form

$$\cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus A_{m, n+1-|I|}) \quad (3.2)$$

are hyperbolic, where  $A_{m, n+1-|I|}$  is a set of at most  $m$  star-subspaces coming from the family of hyperplanes  $\{\cap_{i \in I} H_i \cap H_j\}_{j \in J}$  in the  $(n + 1 - |I|)$ -dimensional projective space  $\cap_{i \in I} H_i \cong \mathbb{P}^{n+1-|I|}(\mathbb{C})$ .

*Proof.* Suppose on the contrary that there exists an entire curve  $f: \mathbb{C} \rightarrow \cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus A_{m, n+1-|I|})$ . Since each star-subspace in  $A_{m, n+1-|I|}$  is constructed from at most  $n + 1 - |I|$  hyperplanes in the family  $\{\cap_{i \in I} H_i \cap H_j\}_{j \in J}$ , the curve  $f$  must avoid completely at least  $|J| - m(n + 1 - |I|)$  hyperplanes in the projective space  $\cap_{i \in I} H_i \cong \mathbb{P}^{n+1-|I|}(\mathbb{C})$ . By the elementary estimate

$$\begin{aligned} |J| - m(n + 1 - |I|) &= q + 1 - 2|I| - m(n - |I|) \\ &\geq 2(n + 1 - |I|) + 1 + \left[ \left( \frac{n+3}{2} \right)^2 - 2(n + 1) - (|I| - 1)(n - |I|) \right] \\ &\geq 2(n + 1 - |I|) + 1, \end{aligned}$$

and by using Theorem 2.1, we derive a contradiction.  $\square$

*Deformation lemma.* For  $2 \leq l \leq n$ , let  $\Delta_l$  be a finite collection of subspaces of dimension  $n+1-l$  coming from the family  $\{H_i\}_{1 \leq i \leq q}$ , possibly with  $\Delta_l = \emptyset$ , and let  $D_l \notin \Delta_l$  be another subspace of dimension  $n+1-l$ , defined as  $D_l = \cap_{i \in I_{D_l}} H_i$ . For an arbitrary hypersurface  $S = \{s = 0\}$  in general position with respect to the family  $\{H_i\}_{1 \leq i \leq q}$  and for  $\epsilon \neq 0$ , we set

$$S_\epsilon = \{\epsilon s + \prod_{i \notin I_{D_l}} h_i^{n_i} = 0\},$$

where  $n_i \geq 1$  are chosen (freely) so that  $\sum_{i \notin I_{D_l}} n_i = q$ . Then the hypersurface  $S_\epsilon$  is also in general position with respect to  $\{H_i\}_{1 \leq i \leq q}$ . We denote by  $\overline{\Delta}_l$  the family of all subspaces of dimension  $n+1-l$  ( $2 \leq l \leq n+1$ ), with the convention  $\overline{\Delta}_{n+1} = \emptyset$ . We shall apply inductively the following lemma.

**Lemma 3.2.** Assume that all complements of the form

$$\cap_{i \in I} H_i \setminus \left( \cup_{j \in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|I|}) \right) \quad (3.3)$$

are hyperbolic where  $I$  and  $J$  are two disjoint subsets of the index set  $\{1, \dots, q\}$  such that  $1 \leq |I| \leq n-1$ , and  $|J| = q + m + 1 - 2|I|$  with some  $0 \leq m \leq |I| - 1$ , and where  $A_{m,n+1-|I|}$  is a set of at most  $m$  star-subspaces coming from the family of hyperplanes  $\{\cap_{i \in I} H_i \cap H_j\}_{j \in J}$  in  $\cap_{i \in I} H_i \cong \mathbb{P}^{n+1-|I|}(\mathbb{C})$ . Then all complements of the form

$$\cap_{i \in I} H_i \setminus \left( \cup_{j \in J} H_j \setminus (((\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon) \cup A_{m,n+1-|I|}) \right) \quad (3.4)$$

are also hyperbolic for sufficiently small  $\epsilon \neq 0$ .

*Proof.* By the definition of  $S_\epsilon$ , we see that  $S_\epsilon \cap (\cap_{m \in M} H_m) = S \cap (\cap_{m \in M} H_m)$  when  $M \cap (\mathbf{Q} \setminus I_{D_l}) \neq \emptyset$ , hence

$$(\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon = ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup (D_l \cap S_\epsilon).$$

When  $|I| \geq l$ , using this, we observe that the two complements (3.3), (3.4) coincide.

Assume therefore  $|I| \leq l-1$ . Suppose by contradiction that there exists a sequence of entire curves  $f_{\epsilon_k}(\mathbb{C})$ ,  $\epsilon_k \rightarrow 0$ , contained in the complement (3.4) for  $\epsilon = \epsilon_k$ . By the Brody Lemma, we may assume that  $(f_{\epsilon_k})$  converges to an entire curve  $f(\mathbb{C}) \subset \cap_{i \in I} H_i$ . We are going to prove that the curve  $f(\mathbb{C})$  lands in some complement of the form (3.3).

Let  $\cap_{k \in K} H_k$  be the smallest subspace containing  $f(\mathbb{C})$ , so that  $I$  is a subset of  $K$ . Take an index  $j$  in  $J \setminus K$ . By stability of intersections, we have

$$\begin{aligned} f(\mathbb{C}) \cap H_j &\subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \\ &\subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|I|} \cup \lim(D_l \cap S_{\epsilon_k}). \end{aligned} \quad (3.5)$$

If the index  $j$  does not belong to  $I_{D_l}$ , then  $H_j \cap D_l \cap S_{\epsilon_k} \subset \overline{\Delta}_{l+1} \cap S$ . It follows from (3.5) that

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|I|}. \quad (3.6)$$

If the index  $j$  belongs to  $I_{D_l}$ , noting that  $\lim(D_l \cap S_{\epsilon_k})$  is contained in  $D_l \cap (\cup_{i \notin I_{D_l}} H_i)$ , hence from (3.5)

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|I|} \cup (D_l \cap (\cup_{i \notin I_{D_l}} H_i)). \quad (3.7)$$

Assume first that  $K = I$ . We claim that (3.6) also holds when the index  $j \in J \setminus I$  belongs to  $I_{D_l}$ . Indeed, for the supplementary part in (3.7), we have

$$f(\mathbb{C}) \cap H_j \cap (D_l \cup_{i \notin I_{D_l}} H_i) \subset \cup_{i \notin I_{D_l}} (f(\mathbb{C}) \cap H_j \cap H_i),$$

so that (3.6) applies here to all  $i \notin I_{D_l}$ . Hence, the curve  $f(\mathbb{C})$  lands inside

$$\cap_{i \in I} H_i \setminus \left( \cup_{j \in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|I|}) \right),$$

contradicting the hypothesis.

Assume now that  $I$  is a proper subset of  $K$ . Let us set

$$A_{m,n+1-|I|,K} = \{X \cap (\cap_{k \in K} H_k) \mid X \in A_{m,n+1-|I|}\}.$$

This set consists of star-subspaces of  $\cap_{k \in K} H_k \cong \mathbb{P}^{n+1-|K|}(\mathbb{C})$ . Let  $B_{m,K}$  be the subset of  $A_{m,n+1-|I|,K}$  containing all star-subspaces of dimension  $n - |K|$  (i.e. of codimension 1 in  $\cap_{k \in K} H_k$ ), and let  $C_{m,K}$  be the remaining part. A star-subspace in  $B_{m,K}$  is of the form  $(\cap_{k \in K} H_k \cap H_j)^*$  for some index  $j \in J \setminus K$ . Let then  $R$  denote the set of such indices  $j$ , so that

$$|R| = |B_{m,K}|.$$

We consider two cases separately, depending on the dimension of the subspace  $Y = \cap_{k \in K} H_k \cap D_l$ .

**Case 1:**  $Y$  is a subspace of dimension  $n - |K|$ . In this case,  $Y$  is of the form  $(\cap_{k \in K} H_k) \cap H_y$  for some index  $y$  in  $I_{D_l}$ . It follows from (3.5), (3.6), (3.7) that the curve  $f(\mathbb{C})$  lands inside

$$\cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K}) \right).$$

To conclude that this set is of the form (3.3), we need to show that

- (1)  $|(J \setminus K) \setminus (R \cup \{y\})| = q + m' + 1 - 2|K|$  with  $|C_{m,K}| \leq m' \leq |K| - 1$ ;
- (2)  $|K| \leq n - 1$ .

Consider (1). We need to verify the corresponding required inequality between cardinalities

$$|C_{m,K}| \leq |(J \setminus K) \setminus (R \cup \{y\})| - q + 2|K| - 1 \leq |K| - 1.$$

The right inequality is equivalent to

$$|(J \setminus K) \setminus (R \cup \{y\})| \leq |\{1, \dots, q\} \setminus K|,$$

which is trivial. The left inequality follows from the elementary estimates

$$\begin{aligned} |(J \setminus K) \setminus (R \cup \{y\})| - q + 2|K| - 1 &\geq |J \setminus K| - |B_{m,K}| - q + 2|K| - 2 \\ &= |J| - |J \cap K| - |B_{m,K}| - q + 2|K| - 2 \\ &= (m - |B_{m,K}|) + (2|K| - 2|I| - |J \cap K| - 1) \\ &\geq |C_{m,K}|, \end{aligned}$$

where the last inequality holds because  $I$  and  $J$  are two disjoint sets and  $I$  is a proper subset of  $K$ .

Consider (2). Suppose on the contrary that  $|K| = n$ . Since  $S$  is in general position with respect to  $\{H_i\}_{1 \leq i \leq 2n+2}$ , we see that

$$\cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K}) \right) = \cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus C_{m,K} \right).$$

Since  $|(J \setminus K) \setminus (R \cup \{y\})| \geq q + 1 - 2n + |C_{m,K}| \geq 3 + |C_{m,K}|$ , the curve  $f$  lands in a complement of at least 3 points in a line. By Picard's Theorem,  $f$  is constant, which is a contradiction.

**Case 2:**  $Y$  is a subspace of dimension at most  $n - |K| - 1$ . In this case, the curve  $f(\mathbb{C})$  lands inside

$$\cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus R} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K} \cup Y^*) \right),$$

which is also of the form (3.3), since

$$|(J \setminus K) \setminus R| \geq q - 2|K| + 1 + |C_{m,K} \cup Y^*|,$$

and since  $|K| \leq n - 1$ , by similar arguments as in **Case 1**.

The Lemma is thus proved. □

*Inductive deformation process and end of the proof of the Main Theorem.* We may begin by applying Lemma 3.2 for  $l = n$  (with  $\overline{\Delta}_{n+1} = \emptyset$ ), firstly with  $\Delta_n = \emptyset$ , and with some  $D_n \in \overline{\Delta}_n$ , since  $(\Delta_n \cup \overline{\Delta}_{n+1}) \cap S = \emptyset$ , hence the assumption of this lemma holds by the Starting Lemma. Next, we reapply Lemma 3.2 inductively until we exhaust all  $D_n \in \overline{\Delta}_n$ . We get at the end a hypersurface  $S_1$  such that all complements of the forms

$$\begin{aligned} \cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus (S_1 \cup A_{m,n+1-|I|})) & \quad (|I|=n-1) \\ \cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus ((\overline{\Delta}_n \cap S_1) \cup A_{m,n+1-|I|})) & \quad (|I| \leq n-2) \end{aligned}$$

are hyperbolic, since when  $|I| = n - 1$ , two components  $\cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus ((\overline{\Delta}_n \cap S_1) \cup A_{m,n+1-|I|}))$  and  $\cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus (S_1 \cup A_{m,n+1-|I|}))$  are equal. Considering this as the starting point of the second step, we apply inductively Lemma 3.2 for  $l = n - 1$  and receive at the end a hypersurface  $S_2$  such that all complements of the forms

$$\begin{aligned} \cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus (S_2 \cup A_{m,n+1-|I|})) & \quad (n-2 \leq |I| \leq n-1) \\ \cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus ((\overline{\Delta}_{n-1} \cap S_2) \cup A_{m,n+1-|I|})) & \quad (|I| \leq n-3) \end{aligned}$$

are hyperbolic, for the same reason as in above. Continuing this process, we get at the end of the  $(n - 1)^{\text{th}}$  step a hypersurface  $S = S_{n-1}$  such that all complements of the forms

$$\cap_{i \in I} H_i \setminus (\cup_{j \in J} H_j \setminus (S_{n-1} \cup A_{m,n+1-|I|})) \quad (1 \leq |I| \leq n-1)$$

are hyperbolic. In particular, by choosing  $m = |I| - 1$ , whence  $|J| = q - |I|$ , and by choosing  $A_{m,n+1-|I|} = \emptyset$ , all complements of the form (3.1) are hyperbolic for  $S = S_{n-1}$ .  $\square$

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